

# PRECISE ASYMPTOTIC APPROXIMATIONS FOR THE KERNELS CORRESPONDING TO THE LÉVY PROCESSES

SIHUN JO AND MINSUK YANG

**ABSTRACT.** Using complex analysis techniques we obtain precise asymptotic approximations for the kernels corresponding to the symmetric  $\alpha$ -stable processes and their fractional derivatives. We apply our method to general Lévy processes whose characteristic functions are radial and satisfy some regularity and size conditions.

## 1. INTRODUCTION

A stochastic process  $\{X_t : t \geq 0\}$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is said to be a Lévy process if  $X_0 = 0$  almost surely, for any  $s, t \geq 0$ , the increment  $X_{t+s} - X_t$  is independent of  $X_u$  for  $0 \leq u \leq t$  and has the same law as  $X_s$ , and  $X_t$  is almost surely right continuous with left limits. By the definition, its characteristic function is given by

$$\mathbb{E}(\exp(i\xi \cdot X_t)) = \exp(-t\eta(\xi)),$$

where  $\eta$  is a Lévy symbol. The symbol  $\eta$  plays a central role. Actually, it is possible to characterize all Lévy processes by their Lévy symbol  $\eta$ , in the sense that, one can construct a Lévy process from any symbol of the form given in the Lévy–Khinchine representation

$$\eta(\xi) = \xi \cdot A\xi - ib \cdot \xi - \int_{\mathbb{R}^d} (e^{i\xi \cdot x} - 1 - i\xi \cdot x 1_{\{|x| < 1\}}(x)) \nu(dx),$$

where  $A$  is a nonnegative-definite symmetric matrix,  $b \in \mathbb{R}^d$ , and the Lévy measure  $\nu$  is a positive Borel measure satisfying

$$\int_{\mathbb{R}^d \setminus \{0\}} (|x|^2 \wedge 1) \nu(dx) < \infty.$$

Here, we use the notation  $a \wedge b = \min\{a, b\}$ . We note that the index of  $\eta$  given by

$$\alpha = \inf \left\{ \lambda > 0 : \lim_{|\xi| \rightarrow \infty} \frac{\eta(\xi)}{|\xi|^\lambda} = 0 \right\}$$

should satisfy  $0 \leq \alpha \leq 2$  (see Blumenthal-Gettoor [BG61]).

Let us mention some of the fundamental symbols which are radial. Brownian motion is a basic example which lies at the source of the theory of stochastic processes. A symmetric  $\alpha$ -stable process is another important example whose symbol is given by

$$\eta(\xi) = |\xi|^\alpha, \quad 0 < \alpha < 2.$$

---

2010 *Mathematics Subject Classification.* Primary 60E07, Secondary 60E10.

*Key words and phrases.* Asymptotic expansion, symmetric stable process, fractional Laplacian, Mellin transform.

We note that the infinitesimal generator of a symmetric  $\alpha$ -stable process  $X_t$  is the fractional Laplacian  $(-\Delta)^{\alpha/2}$  that can be written in the following principal value integral

$$(-\Delta)^{\alpha/2}u(x) = C \text{ p. v. } \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x - y|^{d+\alpha}} dy$$

for some constant  $C$ . Recently there has been many interests from the theory of probability, PDE and SPDE to study such fractional Laplacians (see e.g., Caffarelli-Silvestre [CS07], Chen-Kim-Song [CKS10], Bogdan-Sztonyk [BS07]). We now mention one more example. A relativistic stable processes whose symbol is given by

$$\eta(\xi) = (|\xi|^2 + m^2)^{\alpha/2} - m^\alpha$$

is an example of radial symbol, but it has no scaling property for  $m \neq 0$ . Note that when  $m = 0$ , this coincides to the symmetric  $\alpha$ -stable process. Recently, sharp kernel estimates for relativistic stable processes in open sets are obtained by Chen-Kim-Song [CKS12].

Analytic tools such as semigroup, resolvent and infinitesimal generator have deep connections with the probabilistic properties such as transience and recurrence, asymptotic behavior at infinity, etc. The family of convolution operators indexed by  $t \geq 0$  and given by

$$P_t f(x) = \int_{\mathbb{R}^d} f(x + y) \mathbb{P}(X_t \in dy)$$

is a Markov semigroup. Taking the Fourier transform, we write

$$\mathcal{F}(P_t f)(\xi) = \mathbb{E} \int_{\mathbb{R}^d} e^{i\xi \cdot x} f(X_t + x) dx = e^{-t\eta(-\xi)} \mathcal{F}f(\xi).$$

Thus, it is very important to obtain sharp estimates for kernels  $P_t(x)$  given by

$$P_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-t\eta(-\xi)} d\xi, \quad \beta \geq 0.$$

For more information about Lévy processes, see Sato's monograph [Sat99].

In this article, we propose a method to obtain the asymptotic expansion for  $0 < \alpha < 2$

$$P_t^\alpha(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-t|\xi|^\alpha} d\xi$$

and its fractional derivatives for  $\beta \geq 0$

$$(-\Delta)^{\beta/2} P_t^\alpha(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} |\xi|^\beta e^{-t|\xi|^\alpha} d\xi.$$

Interestingly, for the latter case the decay order does not vary continuously on  $\beta \in \mathbb{R}$ . We also apply our method to the general radial symbol  $\eta$  which has some regularity conditions similar to the standard symbol class in the theory of pseudo-differential operators. There are many results about asymptotic behavior of stable laws. The asymptotic expansion of one-dimensional stable laws was obtained by Linnik [Lin54], Skorohod [Sko54], Zolotarev [Zol86], and others. There are few result for the asymptotic expansion of stable laws in dimension  $d$  greater than 1. We should mention the work of Kolokoltsov [Kol00] that is similar to ours. He used special functions heavily such as Whittaker functions and then he generalized from his results to a different direction compared to ours. We try to use effectively the deep connection between the decay of the kernel and the singularities of the Mellin transform of the symbol. Our method of proof is elementary and direct. It is based

on the very basic but decisive complex analysis technique, that is Cauchy's Residue Theorem with shifting contour integration. As a byproduct, exact constants are also obtained. The Bessel functions naturally arise from the spherical symmetry and we will use some of its basic properties.

The organization of this paper is as follows. In Section 2 we introduce the Mellin transform and its inversion formula and give an important example which is the gamma function. Some of the basic properties of the gamma functions and Bessel functions are also presented for our purpose. In Section 3 we prove complete asymptotics of the kernel corresponding to the symmetric  $\alpha$ -stable processes via Mellin's transform and give some corollaries. The proofs in this section show that the asymptotic behavior of an oscillatory integral is closely connected with the singularities of the Mellin transform of that integral. In Section 4, by refining the same techniques already developed in the previous section, we investigate the asymptotic behavior of the kernels corresponding to more general radial symbol  $\eta$  under some weak conditions.

## 2. PRELIMINARIES

Throughout this article we shall be working on  $\mathbb{R}^d$  with  $d \geq 2$ . Given an integrable function  $f(x)$  on  $\mathbb{R}^d$ , we define its Fourier transform  $\mathcal{F}f(\xi)$  by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} f(x) dx,$$

where  $\xi \in \mathbb{R}^d$  and  $\xi \cdot x = \sum_{j=1}^d \xi_j x_j$ . Similarly the inverse Fourier transform is defined by

$$\mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(\xi) d\xi.$$

Moreover, if  $f$  and  $\mathcal{F}f$  are uniformly continuous and integrable on  $\mathbb{R}^d$ , then the Fourier inversion formula holds, that is,

$$f(x) = \mathcal{F}^{-1}(\mathcal{F}f)(x).$$

More generally, the space of integrable functions on  $\mathbb{R}^d$  is contained in the space of finite complex measures on  $\mathbb{R}^d$  with the total variation norm via the identification  $f dx = d\mu$ . Thus we can generalize the definition of Fourier transform to measures via

$$\mathcal{F}\mu(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} d\mu(x).$$

Given a function  $g(r)$  on  $[0, \infty)$ , we define its Mellin transform  $\mathcal{M}g(z)$  by

$$\mathcal{M}g(z) = \int_0^\infty g(r) r^{z-1} dr.$$

This integral defines a holomorphic function on a strip in the complex plane. It is closely connected to the Fourier transform. Actually, it can be expressed as a Fourier transform. To see this, let  $z = a + ib$  and then change variables  $r = e^{-x}$  to obtain

$$\mathcal{M}g(a + ib) = \int_{-\infty}^\infty g(e^{-x}) e^{-ax} e^{-ibx} dx.$$

Mellin's inversion formula also follows from the Fourier inversion formula. To see this, if  $g(e^{-x})e^{-ax} \in L^1(\mathbb{R}, dx)$  and  $\mathcal{M}g(a + ib) \in L^1(\mathbb{R}, db)$ , then by the Fourier inversion formula

$$g(e^{-x})e^{-ax} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixb} \mathcal{M}g(a - ib) db.$$

Hence we get the Mellin inversion formula

$$(1) \quad g(r) = \frac{1}{2\pi} \int_{\Re(z)-i\infty}^{\Re(z)+i\infty} \mathcal{M}g(z) r^{-z} dz$$

by the change of variables. From now, we shall use the notation

$$\int_{(c)} f(z) dz = \int_{c-i\infty}^{c+i\infty} f(z) dz.$$

A basic example is the gamma function which is defined by the Mellin transform of  $e^{-t}$ . It is initially defined for  $\Re(z) > 0$  by the absolutely convergent integral

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt.$$

We now present some of basic properties of the gamma function. Integrating by parts gives the functional equation  $\Gamma(z+1) = z\Gamma(z)$  and this relation produces a meromorphic continuation. We notice that there is another method to obtain the meromorphic continuation. splitting the integral and expanding  $e^{-t}$  in a power series also produces

$$\begin{aligned} \Gamma(z) &= \int_0^1 \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} t^{z-1} dt + \int_1^{\infty} e^{-t} t^{z-1} dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t} t^{z-1} dt. \end{aligned}$$

The series defines a meromorphic function with simple poles at  $z = 0, -1, -2, \dots$  and the integral on the right defines an entire function. Thus, the gamma function has an analytic continuation to a meromorphic function on  $\mathbb{C}$  whose singularities are simple poles and the residue at  $z = -n$  is  $(-1)^n/n!$ . From Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

$1/\Gamma(z)$  is an entire function with simple zeros at  $z = 0, -1, -2, \dots$  and it vanishes nowhere else.

The Bessel function of the first kind of order  $\nu$  can be defined by a number of different ways. One of the usual ways is the Poisson representation formula which is given by

$$J_{\nu}(r) = \frac{(r/2)^{\nu}}{\Gamma(\nu+1/2)\Gamma(1/2)} \int_{-1}^1 e^{irs} (1-s^2)^{\nu-1/2} ds,$$

where  $\Re(\nu) > -1/2$  and  $r \geq 0$ . Finally, we note that the Mellin transform of  $r^{-\nu} J_{\nu}(r)$  is given by

$$(2) \quad \int_0^{\infty} r^{-\nu} J_{\nu}(r) r^{z-1} dr = \frac{2^{z-\nu-1} \Gamma(\frac{1}{2}z)}{\Gamma(\nu - \frac{1}{2}z + 1)}$$

for  $\nu > -1/2$  and for  $0 < \Re(z) < \Re(\nu) + 3/2$ . We shall use this identity frequently. For more information about Bessel functions, see Watson's monograph [Wat95].

### 3. SYMMETRIC $\alpha$ -STABLE PROCESSES

We begin by considering the classical heat kernel

$$P_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-t|\xi|^2} d\xi.$$

We note that it has the following scaling property

$$P_t(x) = t^{-d/2} P_1(t^{-1/2}x).$$

This reflects selfsimilarity of the Brownian motion, that is, any change of time scale for the Brownian motion has the same effect as some change of spatial scale. Thus we may assume  $t = 1$  and use Fubini's theorem to obtain

$$P_1(x) = (2\pi)^{-d} \prod_{n=1}^d \int_{-\infty}^{\infty} e^{-ix_n \xi_n} e^{-\xi_n^2} d\xi_n.$$

The function  $e^{-z^2}$  is holomorphic in the entire complex plane and has rapid decay as  $|\Re(z)| \rightarrow \infty$ . Hence by Cauchy's theorem we can shift the contour integration to yield

$$\int_{-\infty}^{\infty} e^{-ix_n \xi_n} e^{-\xi_n^2} d\xi_n = e^{-x_n^2/4} \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi} e^{-x_n^2/4}.$$

Therefore

$$P_t(x) = (4\pi t)^{-d/2} e^{-|x|^2/4t}.$$

We now consider the symmetric  $\alpha$ -stable processes

$$P_t^\alpha(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-t|\xi|^\alpha} d\xi.$$

The same argument breaks down in this case because its symbol has no multiplicative structure and has singularity at the origin. However, to prove the following theorem, we shall effectively use the above fundamental technique, that is, to transform the multi-dimensional integral to the contour integration and then to evaluate the integral by shifting the contour.

**Theorem 1.** *Let  $d \geq 2$  and  $0 < \alpha < 2$ . Then  $P_t^\alpha(x)$  has an integral representation*

$$(3) \quad \frac{t^{-d/\alpha}}{\alpha \pi^{d/2}} \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(\frac{z}{\alpha}) \Gamma(\frac{d-z}{2}) 2^{-z}}{\Gamma(\frac{z}{2})} (t^{-1/\alpha} |x|)^{-d+z} dz$$

for  $(d-1)/2 < c < d$  and an asymptotic expansion

$$(4) \quad \frac{t^{-d/\alpha}}{\pi^{d/2}} \sum_{n=1}^{\infty} (1 - 1_{\mathbb{Z}}(n\alpha/2)) \frac{(-1)^n}{n!} \frac{\Gamma(\frac{d+n\alpha}{2}) 2^{n\alpha}}{\Gamma(-\frac{n\alpha}{2})} (t^{-1/\alpha} |x|)^{-d-n\alpha}.$$

*Proof.* We may assume that  $t = 1$  because the kernel has selfsimilarity

$$(5) \quad P_t^\alpha(x) = t^{-d/\alpha} P_1^\alpha(t^{-1/\alpha} x).$$

Claim 1:

$$(6) \quad P_1^\alpha(x) = (2\pi)^{-d/2} |x|^{-d/2+1} \int_0^\infty J_{d/2-1}(|x|r) e^{-r^\alpha} r^{d/2} dr.$$

Proof of Claim 1: In polar coordinates,

$$\begin{aligned} P_1^\alpha(x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-|\xi|^\alpha} d\xi \\ &= (2\pi)^{-d} \int_0^\infty \left( \int_{\mathbb{S}^{d-1}} e^{-ix \cdot r\theta} d\sigma(\theta) \right) e^{-r^\alpha} r^{d-1} dr \\ &= (2\pi)^{-d} \int_0^\infty \mathcal{F}\sigma(-rx) e^{-r^\alpha} r^{d-1} dr, \end{aligned}$$

where  $d\sigma$  is surface measure on  $\mathbb{S}^{d-1}$ . Now (6) is a consequence of the well-known identity

$$\mathcal{F}\sigma(x) = (2\pi)^{d/2} |x|^{-d/2+1} J_{d/2-1}(|x|),$$

where  $J_\nu$  is the Bessel function of the first kind of order  $\nu$ . For the proof and much more general information about Fourier transforms of measures supported on surfaces, we refer to Stein's monograph [Ste93].

Claim 2: For  $(d-1)/2 < c < d$

$$(7) \quad P_1^\alpha(x) = \frac{1}{\alpha\pi^{d/2}} \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(\frac{z}{\alpha}) \Gamma(\frac{d-z}{2}) 2^{-z}}{\Gamma(\frac{z}{2})} |x|^{-d+z} dz.$$

Proof of Claim 2: We recall the gamma function

$$\Gamma(z) = \int_0^\infty e^{-r} r^{z-1} dr,$$

which is the Mellin transform of  $e^{-r}$ . From Mellin's inversion formula (1) we have for  $c > 0$

$$e^{-r} = \frac{1}{2\pi i} \int_{(c)} \Gamma(z) r^{-z} dz.$$

Thus we obtain, by rescaling, the key identity

$$(8) \quad e^{-r^\alpha} = \frac{1}{2\pi i} \int_{(c)} \alpha^{-1} \Gamma(\alpha^{-1}z) r^{-z} dz.$$

Inserting (8) to (6) we see that making the change of variables yields

$$\begin{aligned} P_1^\alpha(x) &= (2\pi)^{-d/2} |x|^{-d/2+1} \int_0^\infty J_{d/2-1}(|x|r) \frac{1}{2\pi i} \int_{(c)} \alpha^{-1} \Gamma(\alpha^{-1}z) r^{-z} dz r^{d/2} dr \\ &= (2\pi)^{-d/2} |x|^{-d/2+1} \frac{1}{2\pi i} \int_{(c)} \alpha^{-1} \Gamma(\alpha^{-1}z) \int_0^\infty J_{d/2-1}(|x|r) r^{-z+d/2} dr dz \\ &= (2\pi)^{-d/2} |x|^{-d} \frac{1}{2\pi i} \int_{(c)} \alpha^{-1} \Gamma(\alpha^{-1}z) \int_0^\infty J_{d/2-1}(r) r^{-z+d/2} dr |x|^z dz. \end{aligned}$$

The identity (2) gives for  $(d-1)/2 < c < d$

$$\int_0^\infty J_{d/2-1}(r) r^{-z+d/2} dr = \frac{2^{-z+d/2} \Gamma(\frac{d-z}{2})}{\Gamma(\frac{z}{2})}.$$

Therefore the integral representation (7) follows.

Claim 3:

$$(9) \quad P_1^\alpha(x) = \frac{1}{\pi^{d/2}} \sum_{n=1}^\infty (1 - 1_{\mathbb{Z}}(n\alpha/2)) \frac{(-1)^n}{n!} \frac{\Gamma(\frac{d+n\alpha}{2}) 2^{n\alpha}}{\Gamma(-\frac{n\alpha}{2})} |x|^{-d-n\alpha}.$$

Proof of Claim 3: To obtain this series representation we consider first the integrand in (7)

$$(10) \quad \frac{\Gamma(\frac{z}{\alpha})\Gamma(\frac{d-z}{2})2^{-z}}{\Gamma(\frac{z}{2})}|x|^{-d+z}.$$

We notice that  $\Gamma(\frac{z}{\alpha})$  is a meromorphic function with simple poles at  $z = 0, -\alpha, -2\alpha, \dots$  and that  $1/\Gamma(\frac{z}{2})$  is an entire function with simple zeros at  $z = 0, -2, -4, \dots$ . So if  $n\alpha/2 \notin \mathbb{Z}$ , then  $z = -n\alpha$  is the pole of the integrand (10).

We define the contour  $\gamma_R$  consists of a rectangle with vertices  $c - iR, c + iR, -\tilde{c} + iR, -\tilde{c} - iR$  with the counterclockwise orientation. Now, we recall the following version of Stirling's formula for the gamma function. If  $a \leq u \leq b$  and  $|v| \rightarrow \infty$ , then

$$(11) \quad |\Gamma(u + iv)| = \sqrt{2\pi}|v|^{u-1/2} \exp\left(-\frac{\pi}{2}|v|\right)\left(1 + O(1/|v|)\right),$$

where the constant implied by  $O$  depends only on  $a$  and  $b$ . By Stirling's formula the integrand (10) has rapid decay as  $|\Im(z)| \rightarrow \infty$ . Hence the two integrals over the horizontal segment on top and bottom tend to 0 as  $R$  tends to infinity. By Cauchy's residue theorem we can shift the vertical line integration to the left  $c \rightarrow -\tilde{c}$  and the remaining integral tends to zero as  $|x| \rightarrow \infty$ . Therefore by letting  $c \rightarrow -\infty$  we obtain the desired expansion (9).

Finally, using the relation (5) we obtain (3) and (4) from (7) and (9).  $\square$

*Remark 1.* This proof shows that the asymptotic behavior of an oscillatory integral is closely connected with the singularities of the Mellin transform of that integral.

*Remark 2.* For the case  $\alpha = 1$  the asymptotic expansion (4) coincides with Poisson's kernel. In fact, we have

$$P_1^1(x) = \frac{1}{\pi^{d/2}|x|^{d+1}} \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}\Gamma(n + \frac{d+1}{2})2^{2n+1}}{(2n+1)!\Gamma(-n - \frac{1}{2})}|x|^{-2n}.$$

A calculation gives

$$(2n+1)!\Gamma(-n-1/2) = 2^{2n+1}n!(-1)^{n+1}\Gamma(1/2)$$

and so we have

$$\begin{aligned} P_1^1(x) &= \frac{1}{\pi^{\frac{d+1}{2}}|x|^{d+1}} \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(n + \frac{d+1}{2})}{n!}|x|^{-2n} \\ &= \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}|x|^{d+1}} \sum_{n=0}^{\infty} \binom{-\frac{d+1}{2}}{n}|x|^{-2n} \\ &= \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}(1 + |x|^2)^{-\frac{d+1}{2}}. \end{aligned}$$

*Remark 3.* For the exceptional case  $\alpha = 2$  the integral identity (3) is still true. So we have

$$P_1^2(x) = \frac{1}{2\pi^{d/2}|x|^d} \frac{1}{2\pi i} \int_{(c)} \Gamma\left(\frac{d-z}{2}\right)2^{-z}|x|^z dz.$$

By Cauchy's residue theorem we can shift the contour integration to the right to obtain

$$\begin{aligned} P_1^2(x) &= \frac{1}{\pi^{d/2}|x|^d} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (|x|/2)^{d+2n} \\ &= (4\pi)^{-d/2} e^{-|x|^2/4}. \end{aligned}$$

We denote  $f \lesssim g$  if  $f \leq cg$  for some constant  $c > 0$  and denote  $f \approx g$  if  $f \lesssim g$  and  $g \lesssim f$ . The following corollary is a direct consequence.

**Corollary 1.** *Let  $d \geq 2$  and  $0 < \alpha < 2$ .*

$$P_t^\alpha(x) \approx t^{-d/\alpha} (1 + t^{-1/\alpha}|x|)^{-(d+\alpha)}.$$

*Proof.* For  $t^{-1/\alpha}|x| > 1$  we have from (4)

$$P_t^\alpha(x) = \frac{-\Gamma(\frac{d+\alpha}{2})2^\alpha}{\pi^{d/2}\Gamma(-\frac{\alpha}{2})} t^{-d/\alpha} (t^{-1/\alpha}|x|)^{-(d+\alpha)} + O\left(\frac{t^2}{|x|^{d+2\alpha}}\right).$$

The result follows from (5).  $\square$

We now consider the symbol

$$\eta(\xi) = |\xi|^\alpha + |\xi|^\beta$$

with  $0 < \alpha < \beta < 2$  that has no scaling property. For this case we have the following interesting corollary which shows the decay order depends mostly on  $\alpha$  for fixed time  $t$  but the scaling relation depends on time. In the next section we study the general symbol of the form

$$\eta(\xi) = |\xi|^\alpha + \eta_1(\xi)$$

under some assumptions on  $\eta_1$ .

**Corollary 2.** *Let  $d \geq 2$ ,  $0 < \alpha < \beta < 2$  and*

$$K_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} \exp(-t[|\xi|^\alpha + |\xi|^\beta]) d\xi.$$

*Then*

$$\begin{cases} K_t(x) \lesssim t^{-d/\beta} (1 + t^{-1/\beta}|x|)^{-(d+\alpha)} & \text{if } t \leq 1 \\ K_t(x) \lesssim t^{-d/\alpha} (1 + t^{-1/\alpha}|x|)^{-(d+\alpha)} & \text{if } t \geq 1. \end{cases}$$

*Proof.* Note that

$$K_t(x) = \int_{\mathbb{R}^d} P_t^\alpha(x-y) P_t^\beta(y) dy.$$

First we assume  $t \leq 1$  and consider the cases  $t^{-1/\beta}|x| \leq 1$  and  $t^{-1/\beta}|x| \geq 1$ . In the case  $t^{-1/\beta}|x| \leq 1$  we have

$$P_t^\beta(y) \lesssim t^{-d/\beta} \leq \frac{2^{d+\alpha} t^{-d/\beta}}{(1 + t^{-1/\beta}|x|)^{d+\alpha}}.$$

Therefore

$$K_t(x) \lesssim \frac{2^{d+\alpha} t^{-d/\beta}}{(1 + t^{-1/\beta}|x|)^{d+\alpha}} \int_{\mathbb{R}^d} P_t^\alpha(x-y) dy.$$



In the case  $t^{-1/\beta}|x| \geq 1$  we split the integral to obtain

$$\begin{aligned} K_t(x) &\lesssim \int_{|y-x| \geq |x|/2} P_t^\alpha(x-y) P_t^\beta(y) dy \\ &\quad + \int_{|y| \geq |x|/2} P_t^\alpha(x-y) P_t^\beta(y) dy \\ &\leq \frac{2^{d+\alpha} t^{-d/\alpha}}{(1+t^{-1/\alpha}|x|)^{d+\alpha}} \int_{|y-x| \geq |x|/2} P_t^\beta(y) dy \\ &\quad + \frac{2^{d+\beta} t^{-d/\beta}}{(1+t^{-1/\beta}|x|)^{d+\beta}} \int_{|y| \geq |x|/2} P_t^\alpha(x-y) dy. \end{aligned}$$

Since  $t^{-1/\beta}|x| \geq 1$  and  $t^{1-\alpha/\beta} \leq 1$ , we have

$$\frac{2^{d+\alpha} t^{-d/\alpha}}{(1+t^{-1/\alpha}|x|)^{d+\alpha}} \leq \frac{2^{d+\alpha} t}{|x|^{d+\alpha}} = \frac{2^{d+\alpha} t^{-d/\beta+1-\alpha/\beta}}{(t^{-1/\beta}|x|)^{d+\alpha}} \leq \frac{4^{d+\alpha} t^{-d/\beta}}{(1+t^{-1/\beta}|x|)^{d+\alpha}}$$

and the required estimate follows.

Now we assume  $t \geq 1$  and consider the cases  $t^{-1/\alpha}|x| \leq 1$  and  $t^{-1/\alpha}|x| \geq 1$ . In the case  $t^{-1/\alpha}|x| \leq 1$  we have

$$P_t^\alpha(x-y) \leq t^{-d/\alpha} \leq \frac{2^{d+\alpha} t^{-d/\alpha}}{(1+t^{-1/\alpha}|x|)^{d+\alpha}}$$

Therefore

$$K_t(x) \lesssim \frac{2^{d+\alpha} t^{-d/\alpha}}{(1+t^{-1/\alpha}|x|)^{d+\alpha}} \int_{\mathbb{R}^d} P_t^\beta(y) dy.$$

In the case  $t^{-1/\alpha}|x| \geq 1$  we split the integral as before to obtain

$$\begin{aligned} K_t(x) &\lesssim \frac{2^{d+\alpha} t^{-d/\alpha}}{(1+t^{-1/\alpha}|x|)^{d+\alpha}} \int_{|y-x| \geq |x|/2} P_t^\beta(y) dy \\ &\quad + \frac{2^{d+\beta} t^{-d/\beta}}{(1+t^{-1/\beta}|x|)^{d+\beta}} \int_{|y| \geq |x|/2} P_t^\alpha(x-y) dy. \end{aligned}$$

Since  $t^{-1/\alpha}|x| \geq 1$  and  $t^{1-\beta/\alpha} \leq 1$ , we have

$$\frac{2^{d+\beta} t^{-d/\beta}}{(1+t^{-1/\beta}|x|)^{d+\beta}} \leq \frac{2^{d+\beta} t}{|x|^{d+\beta}} \leq \frac{4^{d+\beta} t^{-d/\alpha+1-\beta/\alpha}}{(1+t^{-1/\alpha}|x|)^{d+\beta}} \leq \frac{4^{d+\beta} t^{-d/\alpha}}{(1+t^{-1/\alpha}|x|)^{d+\alpha}}$$

and the required estimate follows.  $\square$

Now we are using multi-index notation here and note that formally for multi-index  $\beta$

$$\partial_x^\beta P_t^\alpha(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} (-i\xi)^\beta e^{-t|\xi|^\alpha} d\xi.$$

More generally, the fractional derivatives of  $P_t^\eta(x)$  is defined at least formally by

$$(-\Delta)^{\beta/2} P_t^\alpha(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} |\xi|^\beta e^{-t|\xi|^\alpha} d\xi.$$

Actually, this is the Riesz potential of the probability density function  $P_t^\alpha(x)$ . Sometimes it is necessary to estimate the fractional derivatives of  $P_t^\eta(x)$  especially when one studies a certain type of stochastic partial differential equations. For example, in Chang-Lee [CL12], Sobolev and Besov estimates for an SPDE with a fractional

Laplacian operator was obtained from the decay estimates of  $P_t^\alpha(x)$  and its derivatives. The following corollary is rather surprising.

**Corollary 3.** *Let  $d \geq 2$ ,  $0 < \alpha < 2$  and  $\beta \geq 0$ . Then  $(-\Delta)^{\beta/2} P_t^\alpha(x)$  has an integral representation*

$$\frac{t^{-(d+\beta)/\alpha}}{\alpha \pi^{d/2}} \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(\frac{z}{\alpha}) \Gamma(\frac{d+\beta-z}{2}) 2^{-z}}{\Gamma(\frac{z-\beta}{2})} (t^{-1/\alpha} |x|)^{-d-\beta+z} dz.$$

If  $\beta \neq 0, 2, 4, \dots$ , then as  $|x| \rightarrow \infty$  the first order approximation becomes

$$(-\Delta)^{\beta/2} P_1^\alpha(x) \sim -\frac{2^{\beta-2}}{\pi^{d/2+2}\alpha} \frac{\Gamma(\frac{d+\beta}{2})}{\Gamma(-\frac{\beta}{2})} |x|^{-d-\beta}.$$

If  $\beta = 0, 2, 4, \dots$ , then as  $|x| \rightarrow \infty$  the first order approximation becomes

$$(-\Delta)^{\beta/2} P_1^\alpha(x) \sim -\frac{2^{\beta+\alpha-2}}{\pi^{d/2+2}\alpha} \frac{\Gamma((d+\beta+\alpha)/2)}{\Gamma(-(\beta+\alpha)/2)} |x|^{-d-\beta-\alpha}.$$

*Proof.* We may assume that  $t = 1$  because the kernel has the following scaling property

$$(-\Delta)^{\beta/2} P_t^\alpha(x) = t^{-(d+\beta)/\alpha} (-\Delta)^{\beta/2} P_1^\alpha(t^{-1/\alpha} x).$$

By the same way to derive (6) we also have

$$(-\Delta)^{\beta/2} P_1^\alpha(x) = (2\pi)^{-d/2} |x|^{-d/2+1} \int_0^\infty J_{d/2-1}(|x|r) e^{-r^\alpha} r^{d/2+\beta} dr.$$

Using the identity (2) we have

$$(-\Delta)^{\beta/2} P_1^\alpha(x) = \frac{1}{\alpha \pi^{d/2}} \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(\frac{z}{\alpha}) \Gamma(\frac{d+\beta-z}{2}) 2^{-z+\beta}}{\Gamma(\frac{z-\beta}{2})} |x|^{-d-\beta+z} dz.$$

If  $\beta$  is not an even integer, then 0 is the first pole of the integrand. Similarly, we shift the line of integral for  $s_1$  from  $c_1$  to  $\infty$  so that we obtain

$$(-\Delta)^{\beta/2} P_1^\alpha(x) \sim \frac{2^\beta}{\pi^{d/2}} \frac{\Gamma(\frac{d+\beta}{2})}{\Gamma(-\frac{\beta}{2})} |x|^{-d-\beta}$$

as  $|x| \rightarrow \infty$ .

If  $\beta$  is an even integer, then  $-\alpha$  is the first pole of the integrand. In order to get an asymptotic expansion near infinity, we shift the line of integral for  $z$  from  $c$  to  $\infty$  so that we obtain

$$(-\Delta)^{\beta/2} P_1^\alpha(x) \sim -\frac{2^{\beta+\alpha}}{\pi^{d/2}} \frac{\Gamma((d+\beta+\alpha)/2)}{\Gamma(-(\beta+\alpha)/2)} |x|^{-d-\beta-\alpha}$$

as  $|x| \rightarrow \infty$ . □

*Remark 4.* An examination of the proof shows complete asymptotic expansions can be obtained.

Here is a direct consequence.

**Corollary 4.** *Let  $d \geq 2$ ,  $0 < \alpha < 2$  and  $\beta \geq 0$ . If  $\beta \neq 0, 2, 4, \dots$ , then*

$$(-\Delta)^{\beta/2} P_t^\alpha(x) \approx \left\{ t^{-(d+\beta)/\alpha} \wedge \frac{1}{|x|^{d+\beta}} \right\}.$$

If  $\beta = 0, 2, 4, \dots$ , then

$$(-\Delta)^{\beta/2} P_t^\alpha(x) \approx \left\{ t^{-(d+\beta)/\alpha} \wedge \frac{t}{|x|^{d+\beta+\alpha}} \right\}.$$

*Proof.* Since the function  $|\xi|^\beta e^{-|\xi|^\alpha}$  is integrable on  $\mathbb{R}^d$  for all  $\beta \geq 0$ , the function  $(-\Delta)^{\beta/2} p(1, x)$  is bounded and smooth on  $(0, \infty) \times \mathbb{R}^d$  by the Riemann-Lebesgue lemma.  $\square$

*Remark 5.* In Kim-Kim [KK12], the authors obtained an upper bound

$$|(-\Delta)^{\beta/2} P_1^\alpha(x)| \lesssim |x|^{-(d+\beta)}.$$

This was crucial for them to prove their generalization of a Littlewood-Paley operator for the fractional Laplacian. It was remarked that they thought this upper bound would not be sharp by considering the case  $\beta = 0$ . However, this corollary says that the order of decay is optimal.

#### 4. RADIAL SYMBOLS

In this section we refine our argument in the previous section and give a few of applications which are generalizations of the previous results. We shall study the asymptotics of the kernels corresponding to the radial symbols  $\eta(|\xi|)$  in  $\mathbb{R}^d$ ,  $d \geq 2$ . From now we consider the following general kernel defined by the integral

$$K^\beta(t, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} |\xi|^\beta e^{-t\eta(|\xi|)} d\xi \quad \text{for } \beta \geq 0.$$

By the same way to derive (6), we obtain

$$(12) \quad K^\beta(t, x) = (2\pi)^{-d/2} |x|^{-d/2+1} \int_0^\infty J_{d/2-1}(|x|r) r^{d/2+\beta} e^{-t\eta(r)} dr.$$

Here is our first general result. The key point is to find a Mellin's inversion formula for the symbol  $e^{-t\eta(r)}$  and then to obtain a meromorphic continuation which can be obtained by performing an integration by parts.

**Theorem 2.** *Suppose that  $\eta(r)$  satisfies*

$$(13) \quad \liminf_{r \rightarrow \infty} \frac{\eta(r)}{\log r} = \infty,$$

*and for some  $0 < \alpha < 2$  and  $k > (d+3)/2 + \beta$*

$$(14) \quad \sup_{0 \leq m \leq k} \sup_r |r^{-\alpha+m} D^m \eta(r)| = A < \infty,$$

*where  $D$  denotes the derivative with respect to  $r$ . If  $\beta \neq 0, 2, 4, \dots$ , then for each  $t > 0$*

$$K^\beta(t, x) \sim \frac{2^\beta \Gamma(\frac{d+\beta}{2})}{\pi^{d/2} \Gamma(\frac{-\beta}{2})} \frac{e^{-t\eta(0)}}{|x|^{d+\beta}} \quad \text{as } |x| \rightarrow \infty.$$

*Proof.* Let  $M_t(z)$  denote the Mellin transform

$$M_t(z) = \int_0^\infty e^{-t\eta(r)} r^{z-1} dr.$$

This integral converges absolutely for  $\Re(z) > 0$  by the condition (13).

Claim 1: For  $\Re(z) > 0$

$$(15) \quad M_t(z) = \frac{(-1)^k \Gamma(z)}{\Gamma(z+k)} M_t^k(z),$$

where

$$M_t^k(z) = \int_0^\infty D^k(e^{-t\eta(r)}) r^{z+k-1} dr.$$

Proof of Claim 1: If each of the boundary terms vanishes, then the repeated integration by parts yields

$$\begin{aligned} M_t(z) &= \int_0^\infty e^{-t\eta(r)} r^{z-1} dr \\ &= \frac{-1}{z} \int_0^\infty D(e^{-t\eta(r)}) r^z dr \\ &= \frac{(-1)^2}{(z+1)z} \int_0^\infty D^2(e^{-t\eta(r)}) r^{z+1} dr \\ &\quad \vdots \\ &= \frac{(-1)^k \Gamma(z)}{\Gamma(z+k)} \int_0^\infty D^k(e^{-t\eta(r)}) r^{z+k-1} dr. \end{aligned}$$

To see that the boundary terms vanish, we use the condition (14) to obtain

$$|rD(e^{-t\eta(r)})| = |trD\eta(r)|e^{-t\eta(r)} \leq Atr^\alpha e^{-t\eta(r)}$$

and

$$|r^2 D^2(e^{-t\eta(r)})| \leq |tr^2 D^2\eta(r)|e^{-t\eta(r)} + |t^2 r^2 D\eta(r)|^2 e^{-t\eta(r)} \leq (Atr^\alpha + (Atr^\alpha)^2) e^{-t\eta(r)}.$$

Inductively, we can obtain for  $1 \leq m \leq k$

$$(16) \quad |r^m D^m(e^{-t\eta(r)})| \leq (Atr^\alpha + \dots + (Atr^\alpha)^m) e^{-t\eta(r)}.$$

Consequently we have for  $0 \leq m \leq k$

$$|r^{z+m} D^m(e^{-t\eta(r)})| \rightarrow 0 \quad \text{as } r \rightarrow 0.$$

Using the estimate (16) and the condition (13) we also have

$$|r^{z+m} D^m(e^{-t\eta(r)})| \lesssim r^{\Re(z)+\alpha m} e^{-t\eta(r)} \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

and therefore Claim 1 is proved.

Claim 2: The following Mellin's inversion formula holds for  $c > 0$

$$(17) \quad e^{-t\eta(r)} = \frac{(-1)^k}{2\pi i} \int_{(c)} \frac{\Gamma(z)}{\Gamma(z+k)} M_t^k(z) r^{-z} dz.$$

Proof of Claim 2: If we show that  $M_t(z)$  is integrable, then by Mellin's inversion theorem (1) we get

$$e^{-t\eta(r)} = \frac{1}{2\pi i} \int_{(c)} M_t(z) r^{-z} dz$$

and so we get the result from Claim 1. Using the estimate (16) we have

$$\begin{aligned} |M_t^k(z)| &\leq \int_0^\infty |D^k(e^{-t\eta(r)})r^{z+k-1}|dr \\ &\leq kAt \int_{Atr^\alpha < 1} r^{\Re(z)-1+\alpha} e^{-t\eta(r)} dr \\ &\quad + k(At)^k \int_{Atr^\alpha > 1} r^{\Re(z)-1+\alpha k} e^{-t\eta(r)} dr. \end{aligned}$$

The integrals on the right converge absolutely because of the condition (13). We notice that for  $\Re(z) > -\alpha$  the function  $M_t^k(z)$  is bounded and holomorphic on the bounded strip. Therefore for  $k \geq 2$

$$(18) \quad |M_t(z)| = \left| \frac{(-1)^k \Gamma(z)}{\Gamma(z+k)} M_t^k(z) \right| \lesssim \frac{1}{1 + |\Im(z)|^k}$$

and hence  $M_t(z)$  is integrable.

Claim 3: Then  $K^\beta(t, x)$  has an integral representation for  $(d+1)/2 + \beta < c < d + \beta$

$$(19) \quad \frac{(-1)^k}{\pi^{d/2} |x|^{d+\beta}} \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(z) \Gamma(\frac{-z+d+\beta}{2}) 2^{-z+\beta}}{\Gamma(z+k) \Gamma(\frac{z-\beta}{2})} M_t^k(z) |x|^z dz.$$

Proof of Claim 3: Recall the identity (12), that is

$$K^\beta(t, x) = (2\pi)^{-d/2} |x|^{-d/2+1} \int_0^\infty J_{d/2-1}(|x|r) r^{d/2+\beta} e^{-t\eta(r)} dr.$$

By Claim 2 we obtain

$$K^\beta(t, x) = \frac{(-1)^k}{(2\pi)^{d/2} |x|^{d/2-1}} \frac{1}{2\pi i} \int_0^\infty \int_{(c)} \frac{\Gamma(z)}{\Gamma(z+k)} M_t^k(z) J_{d/2-1}(|x|r) r^{-z+d/2+\beta} dz dr.$$

We now recall that for  $\nu > -1/2$  and  $r \geq 0$

$$(20) \quad |J_\nu(r)| \lesssim (r^\nu \wedge r^{-1/2}).$$

We choose  $(d+1)/2 + \beta < c < d + \beta$  so that the integrand

$$\frac{\Gamma(z)}{\Gamma(z+k)} M_t^k(z) J_{d/2-1}(|x|r) r^{-z+d/2+\beta}$$

is integrable from the estimates (18) and (20). So we can apply Fubini's theorem to obtain

$$K^\beta(t, x) = \frac{(-1)^k}{(2\pi)^{d/2} |x|^{d/2-1}} \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(z)}{\Gamma(z+k)} M_t^k(z) \int_0^\infty J_{d/2-1}(|x|r) r^{-z+d/2+\beta} dr dz.$$

Since we have

$$\begin{aligned} \int_0^\infty J_{d/2-1}(|x|r) r^{-z+d/2+\beta} dr &= |x|^{z-d/2-\beta-1} \int_0^\infty r^{-d/2+1} J_{d/2-1}(r) r^{-z+d+\beta-1} dr \\ &= |x|^{z-d/2-\beta-1} \frac{2^{-z+d/2+\beta} \Gamma(\frac{-z+d+\beta}{2})}{\Gamma(\frac{z-\beta}{2})} \end{aligned}$$

using the identity (2), Claim 3 follows.

Claim 4: As  $|x| \rightarrow \infty$ ,

$$(21) \quad K^\beta(t, x) \sim \frac{2^\beta \Gamma(\frac{d+\beta}{2})}{\pi^{d/2} \Gamma(\frac{-\beta}{2})} \frac{e^{-t\eta(0)}}{|x|^{d+\beta}}.$$

Proof of Claim 4: In the Proof of Claim 2, we noticed that the function  $M_t^k(z)$  is holomorphic for  $\Re(z) > -\alpha$ . Since  $\Gamma((-z+d+\beta)/2)$  has no pole for  $\Re(z) < d+\beta$  and  $\beta$  is not an even integer, the integrand

$$(22) \quad \frac{\Gamma(z)\Gamma(\frac{-z+d+\beta}{2})2^{-z+\beta}}{\Gamma(z+k)\Gamma(\frac{z-\beta}{2})}M_t^k(z)|x|^z$$

in the equation (19) is meromorphic in  $\Re(z) > -\alpha$  and has a simple pole at  $z = 0$ . We calculate the residue at  $z = 0$  as follows:

$$\begin{aligned} \frac{\Gamma(\frac{d+\beta}{2})2^\beta}{\Gamma(k)\Gamma(\frac{-\beta}{2})}M_t^k(0) &= \frac{\Gamma(\frac{d+\beta}{2})2^\beta}{\Gamma(k)\Gamma(\frac{-\beta}{2})} \int_0^\infty D^k(e^{-t\eta(r)})r^{k-1}dr \\ &= \frac{\Gamma(\frac{d+\beta}{2})2^\beta}{\Gamma(\frac{-\beta}{2})}(-1)^{k-1} \int_0^\infty D(e^{-t\eta(r)})dr \\ &= \frac{\Gamma(\frac{d+\beta}{2})2^\beta}{\Gamma(\frac{-\beta}{2})}(-1)^k e^{-t\eta(0)} \end{aligned}$$

by integrating by parts and using the Fundamental Theorem of Calculus. Now we set

$$\tilde{c} = (\alpha \wedge 1)/2$$

and define the contour  $\gamma_R$  consists of a rectangle with vertices  $c - iR, c + iR, -\tilde{c} + iR, -\tilde{c} - iR$  with the counterclockwise orientation. Since the only pole of the integrand (22) inside the rectangle  $\gamma_R$  is at  $z = 0$ , Cauchy's Residue theorem gives

$$\frac{1}{2\pi i} \int_{\gamma_R} \frac{\Gamma(z)\Gamma(\frac{-z+d+\beta}{2})2^{-z+\beta}}{\Gamma(z+k)\Gamma(\frac{z-\beta}{2})}M_t^k(z)|x|^z dz = \frac{(-1)^k \Gamma(\frac{d+\beta}{2})2^\beta}{\Gamma(\frac{-\beta}{2})}e^{-t\eta(0)}.$$

By Stirling's formula, the two integrals over the horizontal segment on top and bottom tend to 0 as  $R$  tends to infinity. Thus, we can shift the vertical line left to obtain

$$K^\beta(t, x) = \frac{2^\beta \Gamma(\frac{d+\beta}{2})}{\pi^{d/2} \Gamma(\frac{-\beta}{2})} \frac{e^{-t\eta(0)}}{|x|^{d+\beta}} + R(t, x),$$

where

$$R(t, x) = \frac{(-1)^k 2^\beta}{\pi^{d/2} |x|^{d+\beta}} \frac{1}{2\pi i} \int_{(-\tilde{c})} \frac{\Gamma(z)\Gamma(\frac{-z+d+\beta}{2})2^{-z+\beta}}{\Gamma(z+k)\Gamma(\frac{z-\beta}{2})}M_t^k(z)|x|^z dz.$$

Using Stirling's formula, we have the estimate

$$\left| \frac{\Gamma(z)\Gamma(\frac{-z+d+\beta}{2})2^{-z+\beta}}{\Gamma(z+k)\Gamma(\frac{z-\beta}{2})}M_t^k(z)|x|^z \right| \lesssim |\Im(z)|^{-\Re(z)+d/2+\beta-k}.$$

and therefore  $|x|^{d+\beta}R(t, x)$  tends to 0 as  $|x|$  tends to infinity.  $\square$

*Remark 6.* The key point of the proof is that after inserting the Mellin inversion formula (17) to (12) and changing the order of integration, the remaining integrand (22) has meromorphic continuation.

*Remark 7.* If  $1 < \alpha < 2$ , we choose  $\tilde{c} = (\alpha + 1)/2$  so that the integrand (22) has two simple poles at  $z = 0$  and  $z = -1$ . Then the residue at  $z = -1$  produces the second order approximation. Actually, one can go further, if the symbol  $\eta$  satisfies the regularity condition for all  $k \in \mathbb{N}$ , then we can obtain an asymptotic expansion of the kernel.

*Remark 8.* The assumption (14) is analogous to that of standard symbol class in the theory of pseudo-differential operators.

*Remark 9.* The meromorphic continuation of the Mellin transform using resolution of singularities goes back to Bernstein-Gelfand [BG69].

We note that the main contribution to  $K^\beta(t, x)$  comes from the integration near  $r = 0$ . We guess that the assumption (14) could be weakened away from the origin. This idea is made precise in the following proposition.

Let  $\psi(r)$  be a smooth function defined in  $\mathbb{R}$ , with the properties that  $\psi(r) = 0$  for  $r \leq 1$ , and  $\psi(r) = 1$  for  $r \geq 2$ . Let us set for  $\beta \geq 0$

$$\mathcal{E}^\beta(t, x) = \int_1^\infty J_{d/2-1}(|x|r) \psi(r) r^{d/2+\beta} e^{-t\eta(r)} dr.$$

We shall show that this is an error term.

**Proposition 1.** *Suppose that  $\eta(r) \in C^N((0, \infty))$  satisfies the integrability condition*

$$(23) \quad \int_1^\infty r^{d/2-1/2+\beta} e^{-t\eta(r)} dr < \infty.$$

*Then for fixed  $t > 0$*

$$|\mathcal{E}^\beta(t, x)| \lesssim |x|^{-N-1/2}.$$

*Proof.* We begin by recalling that for  $\nu > -1/2$  and  $r > 0$

$$(24) \quad D[r^\nu J_\nu(r)] = r^\nu J_{\nu-1}(r),$$

where  $D$  denotes the derivative with respect to  $r$ . Fix  $t > 0$  and denote

$$g(r) = \psi(r) r^{d/2+\beta} e^{-t\eta(r)}.$$

Then using (24) we have

$$\begin{aligned} & \int_0^\infty J_{\nu-1}(|x|r) g(r) dr \\ &= \int_0^\infty (|x|r)^\nu J_{\nu-1}(|x|r) (|x|r)^{-\nu} g(r) dr \\ &= \int_0^\infty \frac{1}{k} D[(|x|r)^\nu J_\nu(|x|r)] (|x|r)^{-\nu} g(r) dr \\ &= \lim_{r \rightarrow \infty} \frac{1}{k} J_\nu(|x|r) g(r) - \frac{1}{k} \int_0^\infty (|x|r)^\nu J_\nu(|x|r) D[(|x|r)^{-\nu} g(r)] dr \\ &= \frac{1}{k} \int_0^\infty J_\nu(|x|r) r^{-1} (\nu I - rD) g(r) dr \end{aligned}$$

since  $\lim_{r \rightarrow \infty} J_\nu(|x|r) g(r) = 0$ . If we denote

$$L_\nu = r^{-1}(\nu I - rD),$$

then we can write

$$\int_0^\infty J_{\nu-1}(|x|r) g(r) dr = \frac{1}{|x|} \int_0^\infty J_\nu(|x|r) L_\nu g(r) dr.$$

Carrying out the repeated integration by parts gives

$$\mathcal{E}^\beta(t, x) = \frac{1}{|x|^N} \int_0^\infty J_{\nu+N-1}(|x|r) L_{\nu+N-1} L_{\nu+N-2} \cdots L_\nu g(r) dr.$$

Using the decay of Bessel functions (20) and the integrability condition (23), we have

$$\begin{aligned}
|\mathcal{E}^\beta(t, x)| &\lesssim \frac{1}{|x|^{N+1/2}} \int_1^\infty r^{-1/2} |L_{\nu+N-1} L_{\nu+N-2} \cdots L_\nu g(r)| dr \\
&\lesssim \frac{1}{|x|^{N+1/2}} \int_1^\infty r^{-1/2} \sum_{n=0}^N r^{n-N} |D^n g(r)| dr \\
&\lesssim \frac{1}{|x|^{N+1/2}} \sum_{n=0}^N \int_1^\infty r^{-1/2+n-N} \sum_{k=0}^n |D^{n-k} r^{d/2+\beta}| |D^k e^{-t\eta(r)}| dr \\
&\lesssim \frac{1}{|x|^{N+1/2}} \sum_{n=0}^N \sum_{k=0}^n \int_1^\infty r^{-1/2-N+d/2+\beta+k} |D^k e^{-t\eta(r)}| dr \\
&\lesssim \frac{1}{|x|^{N+1/2}}
\end{aligned}$$

and the result follows.  $\square$

We have the same main term under the weak assumptions compare to those of the previous theorem. This shows that the local behavior of the symbol  $\eta$  near  $r = 0$  is important.

**Corollary 5.** *Suppose that  $\eta(r)$  is a real-valued continuous function on  $[0, \infty)$  and that there exist  $0 < \alpha < 2$ ,  $k > (d+3)/2 + \beta$ , and  $M > 0$  such that*

$$(25) \quad \sup_{1 \leq m \leq k} \sup_{0 < r < 1} r^{-\alpha+m} |D^m \eta(r)| < \infty,$$

$$(26) \quad \sup_{0 \leq m \leq k} \sup_{1 < r < \infty} |D^m \eta(r)| r^{-M} < \infty,$$

and

$$(27) \quad \liminf_{r \rightarrow \infty} \frac{\eta(r)}{\log r} = \infty.$$

If  $\beta \neq 0, 2, 4, \dots$ , then for each  $t > 0$

$$K^\beta(t, x) \sim \frac{2^\beta \Gamma(\frac{d+\beta}{2})}{\pi^{d/2} \Gamma(\frac{-\beta}{2})} \frac{e^{-t\eta(0)}}{|x|^{d+\beta}} \quad \text{as } |x| \rightarrow \infty$$

*Proof.* By essentially the same way of in the proof of Theorem 2, we can prove the following Mellin inversion formula for  $c > 0$

$$e^{-t\eta(r)} = \frac{(-1)^k}{2\pi i} \int_{(c)} \frac{\Gamma(z)}{\Gamma(z+k)} M_t^k(z) r^{-z} dz,$$

where

$$M_t^k(z) = \int_0^\infty u^{z+k-1} \left( \frac{d}{du} \right)^k e^{-t\eta(u)} du.$$

The remaining proof is the same and so it is omitted.  $\square$

We consider the perturbation of the symbol corresponding to the symmetric  $\alpha$ -stable process that has the form  $\eta(r) = r^\alpha + \eta_1(r)$  under some conditions of  $\eta_1(r)$ .



**Theorem 3.** *Let  $0 < \alpha < 2$ . Suppose that  $\eta(r) = r^\alpha + \eta_1(r)$  and that  $\eta_1(r)$  is a real-valued continuous function on  $[0, \infty)$  that satisfies for some  $\delta > \alpha$ ,  $k > (d+3)/2 + \beta$ , and  $M > 0$  such that*

$$(28) \quad \sup_{1 \leq m \leq k} \sup_{0 < r < 1} r^{-\delta+m} |D^m \eta_1(r)| < \infty,$$

$$(29) \quad \sup_{0 \leq m \leq k} \sup_{1 < r < \infty} |D^m \eta_1(r)| r^{-M} < \infty,$$

and

$$(30) \quad \liminf_{r \rightarrow \infty} \frac{\eta_1(r)}{\log r} = \infty.$$

If  $\beta = 0, 2, 4, \dots$ , then for each  $t > 0$

$$K^\beta(t, x) \sim -\frac{2^{\beta+\alpha} \Gamma(\frac{d+\beta+\alpha}{2})}{\pi^{d/2} \Gamma(-\frac{\beta+\alpha}{2})} \frac{t e^{-t\eta_1(0)}}{|x|^{d+\beta+\alpha}}.$$

*Proof.* From the identity (12) we already know that

$$(31) \quad K^\beta(t, x) = \frac{1}{(2\pi)^{d/2} |x|^{d+\beta}} \int_0^\infty J_{d/2-1}(r) e^{-t(r/|x|)^\alpha} e^{-t\eta_1(r/|x|)} r^{d/2+\beta} dr.$$

Since  $\Gamma(z)$  is the Mellin transform of  $e^{-r}$ , we have for  $c_0 > 0$

$$(32) \quad e^{-t(r/|x|)^\alpha} = \frac{1}{2\pi i \alpha} \int_{(c_0)} \Gamma\left(\frac{s}{\alpha}\right) (t^{1/\alpha} r/|x|)^{-s} ds$$

by Mellin's inversion formula (1). We also have for  $c > 0$

$$(33) \quad e^{-t\eta_1(r/|x|)} = \frac{(-1)^k}{2\pi i} \int_{(c)} \frac{\Gamma(z)}{\Gamma(z+k)} M_t^k(z) (r/|x|)^{-z} dz,$$

where

$$M_t^k(z) = \int_0^\infty u^{z+k-1} \left(\frac{d}{du}\right)^k e^{-t\eta_1(u)} du.$$

Putting (32) and (33) together with (31), we obtain

$$\begin{aligned} K^\beta(t, x) &= \frac{(-1)^{k+1}}{\alpha(2\pi)^{d/2+2} |x|^{d+\beta}} \int_0^\infty \int_{(c_0)} \int_{(c)} \frac{\Gamma(z) \Gamma(\frac{s}{\alpha})}{\Gamma(z+k)} M_t^k(z) t^{-s/\alpha} |x|^{s+z} \\ &\quad \times J_{d/2-1}(r) r^{-s-z+d/2+\beta} dz ds dr. \end{aligned}$$

From the identity (2) we have

$$\int_0^\infty J_{d/2-1}(r) r^{-s-z+d/2+\beta} dr = \frac{2^{-s-z+d/2+\beta} \Gamma(\frac{-s-z+d+\beta}{2})}{\Gamma(\frac{s+z-\beta}{2})}.$$

We choose  $(d+1)/2 + \beta < c + c_0 < d + \beta$  so that we can apply Fubini's theorem and the identity (2) to obtain

$$\begin{aligned}
p^\beta(t, x) &= \frac{(-1)^{k+1}}{\alpha(2\pi)^{d/2+2}|x|^{d+\beta}} \int_{(c_0)} \int_{(c)} \frac{\Gamma(z)\Gamma(\frac{s}{\alpha})}{\Gamma(z+k)} M_t^k(z) t^{-s/\alpha} |x|^{s+z} \\
&\quad \times \int_0^\infty J_{d/2-1}(r) r^{-s+z+d/2+\beta} dr dz ds \\
&= \frac{(-1)^{k+1}}{\alpha(2\pi)^{d/2+2}|x|^{d+\beta}} \int_{(c_0)} \int_{(c)} \frac{\Gamma(z)\Gamma(\frac{s}{\alpha})}{\Gamma(z+k)} M_t^k(z) t^{-s/\alpha} |x|^{s+z} \\
&\quad \times \frac{2^{-s-z+d/2+\beta} \Gamma(\frac{-s-z+d+\beta}{2})}{\Gamma(\frac{s+z-\beta}{2})} dz ds \\
&= \frac{(-1)^{k+1} 2^{d/2+\beta}}{\alpha(2\pi)^{d/2+2}|x|^{d+\beta}} \int_{(c_0)} \int_{(c)} \frac{\Gamma(s/\alpha)\Gamma(z)\Gamma(\frac{-s-z+d+\beta}{2})}{\Gamma(z+k)\Gamma(\frac{s+z-\beta}{2})} \\
&\quad \times M_t^k(z) t^{-s/\alpha} (|x|/2)^{s+z} dz ds.
\end{aligned}$$

Now, we define the contour  $\gamma_R$  consists of a rectangle with vertices  $c - iR, c + iR, -\delta/2 + iR, -\delta/2 - iR$  with the counterclockwise orientation. The integrand

$$\frac{\Gamma(z)\Gamma(\frac{-s-z+d+\beta}{2})}{\Gamma(z+k)\Gamma(\frac{s+z-\beta}{2})} M_t^k(z) (|x|/2)^z$$

has only one simple pole at  $z = 0$  in the rectangle bounded by  $\gamma_R$ . By Cauchy's Residue Theorem,

$$\frac{1}{2\pi i} \int_{\gamma_R} \frac{\Gamma(z)\Gamma(\frac{-s-z+d+\beta}{2})}{\Gamma(z+k)\Gamma(\frac{s+z-\beta}{2})} M_t^k(z) (|x|/2)^z dz = \frac{\Gamma(\frac{-s+d+\beta}{2})}{\Gamma(k)\Gamma(\frac{s-\beta}{2})} M_t^k(0).$$

We have  $M_t^k(0) = (-1)^k \Gamma(k) e^{-t\eta_1(0)}$ . As  $R \rightarrow \infty$ , the two integrals over the line parallel to  $x$ -axis go to zero by Stirling's formula. Thus, we can shift the line integral left to obtain

$$\frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(z)\Gamma(\frac{-s-z+d+\beta}{2})}{\Gamma(z+k)\Gamma(\frac{s+z-\beta}{2})} M_t^k(z) (|x|/2)^{s+z} dz = \frac{(-1)^k \Gamma(\frac{-s+d+\beta}{2})}{\Gamma(\frac{s-\beta}{2})} e^{-t\eta_1(0)} + R_{t,x}(s),$$

where

$$R_{t,x}(s) = \frac{1}{2\pi i} \int_{(-\frac{\alpha+\delta}{2})} \frac{\Gamma(z)\Gamma(\frac{-s-z+d+\beta}{2})}{\Gamma(z+k)\Gamma(\frac{s+z-\beta}{2})} M_t^k(z) (|x|/2)^z dz.$$

Hence we have

$$\begin{aligned}
K^\beta(t, x) &= \frac{-2^{d/2+\beta} i e^{-t\eta_1(0)}}{\alpha(2\pi)^{d/2+1} |x|^{d+\beta}} \int_{(c_0)} \frac{\Gamma(s/\alpha) \Gamma(\frac{-s+d+\beta}{2})}{\Gamma(\frac{s-\beta}{2})} t^{-s/\alpha} (|x|/2)^s ds \\
&\quad + \frac{(-1)^{k+1} 2^{d/2+\beta} i}{\alpha(2\pi)^{d/2+1} |x|^{d+\beta}} \int_{(c_0)} \Gamma(s/\alpha) R_{t,x}(s) t^{-s/\alpha} |x|^s ds \\
&= -\frac{2^{\beta+\alpha} \Gamma(\frac{d+\beta+\alpha}{2})}{\pi^{d/2} \Gamma(-\frac{\beta+\alpha}{2})} \frac{t e^{-t\eta_1(0)}}{|x|^{d+\beta+\alpha}} \\
&\quad + \frac{-2^{d/2+\beta} i e^{-t\eta_1(0)}}{\alpha(2\pi)^{d/2+1} |x|^{d+\beta}} \int_{(-\frac{3}{2}\alpha)} \frac{\Gamma(s/\alpha) \Gamma(\frac{-s+d+\beta}{2})}{\Gamma(\frac{s-\beta}{2})} t^{-s/\alpha} (|x|/2)^s ds \\
&\quad + \frac{(-1)^{k+1} 2^{d/2+\beta} i}{\alpha(2\pi)^{d/2+1} |x|^{d+\beta}} \int_{(\frac{\delta-\alpha}{4})} \Gamma(s/\alpha) R_{t,x}(s) t^{-s/\alpha} |x|^s ds.
\end{aligned}$$

For  $k > (d+3)/2 + \beta$ , the last two integrals converge absolutely by Stirling's formula and they are bounded by

$$|x|^{-d-\beta-(\frac{3\alpha+\delta}{4} \wedge \frac{3}{2}\alpha)},$$

as  $|x| \rightarrow \infty$ . Since  $(\frac{3\alpha+\delta}{4} \wedge \frac{3}{2}\alpha) > \alpha$ , the theorem is proved.  $\square$

#### ACKNOWLEDGEMENTS

We thank Zhen-Qing Chen for his helpful comments and for giving related references during the summer school in Probability 2012 in Republic of Korea.

#### REFERENCES

- [BG61] R. M. Blumenthal and R. K. Gettoor. Sample functions of stochastic processes with stationary independent increments. *J. Math. Mech.*, 10:493–516, 1961.
- [BG69] I. N. Bernšteĭn and S. I. Gel'fand. Meromorphy of the function  $P^\lambda$ . *Funkcional. Anal. i Priložen.*, 3(1):84–85, 1969.
- [BS07] Krzysztof Bogdan and Paweł Sztonyk. Estimates of the potential kernel and Harnack's inequality for the anisotropic fractional Laplacian. *Studia Math.*, 181(2):101–123, 2007.
- [CKS10] Zhen-Qing Chen, Panki Kim, and Renming Song. Heat kernel estimates for the Dirichlet fractional Laplacian. *J. Eur. Math. Soc. (JEMS)*, 12(5):1307–1329, 2010.
- [CKS12] Zhen-Qing Chen, Panki Kim, and Renming Song. Sharp heat kernel estimates for relativistic stable processes in open sets. *Ann. Probab.*, 40(1):213–244, 2012.
- [CL12] Tongkeun Chang and Kijung Lee. On a stochastic partial differential equation with a fractional Laplacian operator. *Stochastic Process. Appl.*, 122(9):3288–3311, 2012.
- [CS07] Luis Caffarelli and Luis Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.
- [KK12] Ildoo Kim and Kyeong-Hun Kim. A generalization of the Littlewood-Paley inequality for the fractional Laplacian  $(-\Delta)^{\alpha/2}$ . *J. Math. Anal. Appl.*, 388(1):175–190, 2012.
- [Kol00] Vassili Kolokoltsov. Symmetric stable laws and stable-like jump-diffusions. *Proc. London Math. Soc. (3)*, 80(3):725–768, 2000.
- [Lin54] Yu. V. Linnik. On stable probability laws with exponent less than one. *Doklady Akad. Nauk SSSR (N.S.)*, 94:619–621, 1954.
- [Sat99] Ken-iti Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original, Revised by the author.
- [Sko54] A. V. Skorohod. Asymptotic formulas for stable distribution laws. *Dokl. Akad. Nauk SSSR (N.S.)*, 98:731–734, 1954.
- [Ste93] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press,

- Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.
- [Wat95] G. N. Watson. *A treatise on the theory of Bessel functions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995. Reprint of the second (1944) edition.
- [Zol86] V. M. Zolotarev. *One-dimensional stable distributions*, volume 65 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1986. Translated from the Russian by H. H. McFaden, Translation edited by Ben Silver.

(S. Jo) DEPARTMENT OF MATHEMATICS, YONSEI UNIVERSITY, SEOUL 120-749, REPUBLIC OF KOREA

*E-mail address:* `schevinger@yonsei.ac.kr`

(M. Yang) DEPARTMENT OF MATHEMATICS, YONSEI UNIVERSITY, SEOUL 120-749, REPUBLIC OF KOREA

*E-mail address:* `Yan9@yonsei.ac.kr`